Creating a Recursive Spectral Bisection Using Lanczos Algorithm

Brigham Young University, CS 584
Fall 2000

Computing the Second Eigenvalue and Vector

Given a graph $G$ with nodes $n = 4$,

$$
\begin{array}{c}
0 \\
1 \\
2 \\
3
\end{array}
\begin{array}{c}
0 \\
1 \\
2 \\
3
\end{array}

\begin{array}{ccc}
0 & 1 & 2 \\
1 & 2 & -1 & -1 \\
2 & -1 & 2 & -1 \\
3 & -1 & -1 & 1
\end{array}

the LaPlacian matrix $L(G)$ is computed by $D(G) - A(G)$.

$$
\text{LaPlacian matrix } L(G) =
\begin{bmatrix}
0 & 1 & 2 & 3 \\
0 & 2 & -1 & -1 \\
1 & -1 & 2 & -1 \\
2 & -1 & -1 & 1
\end{bmatrix}

\begin{array}{c}
0 \\
1 \\
2 \\
3
\end{array}
\begin{array}{c}
0 \\
1 \\
2 \\
3
\end{array}

\begin{array}{ccc}
0 & 1 & 2 & 3 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
2 & 1 & 1 & 1 \\
3 & 1
\end{array}

\text{degree matrix } D(G) \quad \text{adjacency matrix } A(G)
$$

Spectral bisection uses the eigenvector $(v_2)$ associated with the second eigenvalue ($\lambda_2$) to split $G$. Therefore, we must compute $\lambda_2$ and $v_2$. We determined the following values by brute force.

$$
\lambda = 0, 1, 3, 4 \quad \lambda_2 = 1 \quad v_2 = \langle -\frac{1}{2}, -\frac{1}{2}, 0, 1 \rangle
$$

Spectral Bisection simply specifies that for each element $j$ in $v_2$, if $v_{2,j} < 0$,
place node \( j \) in partition A and if \( v_{2j} \geq 0 \), place node \( j \) in partition B. The algorithm is then applied recursively \( m \) times to each partition to achieve a partitioning into \( 2^m \) partitions. This yields the partitioning

For graphs larger than \( G \), solving for the exact value of \( \lambda_2 \) by brute force becomes computationally intensive. A better choice is to approximate \( \lambda_2 \) and \( v_2 \) using Lanczos algorithm.

Lanczos algorithm is a heuristic for finding eigenvalues. Because we are only interested in the sign of each element in \( v_2 \), and because the spectral bisection is only a heuristic itself, Lanczos will yield an acceptable approximation of the eigenvalues (\( \lambda \)) to bisect the graph. Lanczos approximates the eigenvalues by finding the eigenvalues of a much smaller graph \( T \). \( T \) has the special property that it is \( k \times k \) symmetric and tridiagonal where \( k \ll n \). Lanczos iteratively increases the dimension of \( T \), beginning with \( k = 0 \). At each iteration, the eigenvalues of \( T \) may be computed. These eigenvalues will be an approximation of a subset of the eigenvalues of \( L \). Fortunately, the first eigenvalues to emerge are the largest and smallest. \( \lambda_2 \) will be the second smallest eigenvalue of \( T \) after \( k \) iterations of Lanczos algorithm.

The following is pseudo code for Lanczos algorithm.

<table>
<thead>
<tr>
<th>line</th>
<th>operation</th>
<th>comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( k = 0 )</td>
<td>the iteration and current dimension of ( T )</td>
</tr>
<tr>
<td>2</td>
<td>choose arbitrary vector ( r )</td>
<td>length= n</td>
</tr>
<tr>
<td>3</td>
<td>vector ( t_0 = O )</td>
<td>length= n</td>
</tr>
<tr>
<td></td>
<td>repeat</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( b_k = norm(r) )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( k = k + 1 )</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>( t_k = r/b_{k-1} )</td>
<td>divide each element of ( r ) by ( b_{k-1} )</td>
</tr>
<tr>
<td>7</td>
<td>( r = (L(G) \times t_k^T)^T )</td>
<td>vector-matrix multiply</td>
</tr>
<tr>
<td>8</td>
<td>( r = r - (b_{k-1} \ast t_{k-1}) )</td>
<td>multiply each element of ( t_{k-1} ) by ( b_{k-1} )</td>
</tr>
<tr>
<td>9</td>
<td>( a_k = t_k \cdot r )</td>
<td>dot product of ( t_k ) and ( r )</td>
</tr>
<tr>
<td>10</td>
<td>( r = r - (a_k \ast t_k) )</td>
<td>multiply each element of ( t_k ) by ( a_k )</td>
</tr>
<tr>
<td>11</td>
<td>until convergence</td>
<td>as ( k \to n ), the ( \lambda ) approach the true ( \lambda ) of ( L ). As this happens, the ( \lambda ) for ( T ) will change more slowly between iterations.</td>
</tr>
</tbody>
</table>
After each iteration \( i \), the matrix \( T \) is the tridiagonal symmetric matrix:

\[
T = \begin{bmatrix}
a_1 & b_1 & & \\
b_1 & a_2 & b_2 & \\
& b_2 & \ddots & \ddots \\
& & \ddots & a_{k-1} & b_{k-1} \\
& & & b_{k-1} & a_k
\end{bmatrix}
\]

A trace of Lanczos for the above graph \( G \) and LaPlacian matrix \( L(G) \):

<table>
<thead>
<tr>
<th>line</th>
<th>( k )</th>
<th>operation</th>
<th>variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>( r = \text{&lt;random&gt;} )</td>
<td>( r = &lt;1, -1, 1, -1&gt; )</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>( t = O )</td>
<td>( t_0 = &lt;0, 0, 0, 0&gt; )</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>( b_0 = \sqrt{1^2 + (-1)^2 + 1^2 + (-1)^2} )</td>
<td>( b_0 = 2 )</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>( k = k + 1 )</td>
<td>( k = 1 )</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>( t_1 = \frac{&lt;1,-1,1,-1&gt;}{2} )</td>
<td>( t_1 = \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} )</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>( r = (L(G) \times \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})^\top )</td>
<td>( r = &lt;1, -2, 2, -1&gt; )</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>( r = &lt;1, -2, 2, -1&gt; \cdot -(2*&lt;0, 0, 0, 0&gt;) )</td>
<td>( r = &lt;1, -2, 2, -1&gt; )</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>( a_1 = &lt;\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}&gt; &lt;1, -2, 2, -1&gt; )</td>
<td>( a_1 = 3 )</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>( r = &lt;1, -2, 2, -1&gt; \cdot -(3*&lt;\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}&gt;) )</td>
<td>( r = &lt;\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}&gt; )</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>( b_1 = \sqrt{(-\frac{1}{2})^2 + (-\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2} )</td>
<td>( b_1 = 1 )</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>( k = k + 1 )</td>
<td>( k = 2 )</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>( t_2 = \frac{&lt;\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}&gt;}{2} )</td>
<td>( t_2 = &lt;\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}&gt; )</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>( r = (L(G) \times \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})^\top )</td>
<td>( r = &lt;-1, -1, 2, 0&gt; )</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>( r = &lt;-1, -1, 2, 0&gt; \cdot -(1*&lt;\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}&gt;) )</td>
<td>( r = &lt;-\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}&gt; )</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>( a_2 = &lt;-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}&gt; &lt;\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}&gt; )</td>
<td>( a_2 = 2 )</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>( r = &lt;-\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{3}{2}&gt; \cdot -(2*&lt;\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}&gt;) )</td>
<td>( r = &lt;-\frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}&gt; )</td>
</tr>
</tbody>
</table>
11 \[ T = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \lambda \approx 1.4, 3.6 \]

11 \[ \lambda \approx 1.4, 3.6 \]

4 \[ b_2 = \sqrt{(-\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 + (-\frac{1}{2})^2} \]

4 \[ b_2 = 1 \]

5 \[ k = k + 1 \]

6 \[ t_3 = \langle -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle \]

6 \[ t_3 = \langle -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle \]

7 \[ r = (L(G) \times \langle -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle) ^\top \]

7 \[ r = \langle 2, 1, 2, -1 \rangle \]

8 \[ r = \langle -2, 1, 2, -1 \rangle \]

8 \[ r = \langle -2, 1, 2, -1 \rangle \]

8 \[ -(1 \times \langle -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle) \]

8 \[ \langle -\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{3}{2} \rangle \]

9 \[ a_3 = \langle -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle \]

9 \[ r = \langle -\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{3}{2} \rangle \]

9 \[ \langle -\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{3}{2} \rangle \]

9 \[ a_3 = 3 \]

10 \[ r = \langle -\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{3}{2} \rangle \]

10 \[ r = \langle -\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{3}{2} \rangle \]

10 \[ r = \langle -\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{3}{2} \rangle \]

10 \[ -(3 \times \langle -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \rangle) \]

10 \[ \langle -\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{3}{2} \rangle \]

10 \[ \langle -\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{3}{2} \rangle \]

10 \[ a_3 = 3 \]

11 \[ \lambda \approx 1, 3, 4 \]

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At this point, our known value for \( \lambda_2 \) has emerged. This was done on a matrix of dimension \( k = n - 1 \). However, as \( n \rightarrow \infty \), the dimension \( k \) of \( T \) becomes approximately 1 order of magnitude smaller than \( n \).

Now, we plug \( \lambda_2 \) back into \( L(G) \) to find \( v_2 \). This, as before, yields \( \langle -\frac{1}{2}, -\frac{1}{2}, 0, 1 \rangle \). If we tried to find \( v_2 \) with respect to \( T \), we would not have each node of \( G \) represented in \( v_2 \).

Again, the partitioning calculated is

![Graph](image-url)